Morgan Sly

Basel Problem

What is the probability that two $(a,b) \in [1,N]^2$ have $\gcd(a,b) = 1$, as $N \to \infty$

Define:

$$S_N := \frac{\#\{(a,b)\in[1,N]^2 : \mathrm{gcd}(a,b)=1\}}{N^2}$$

Inclusion-Exclusion (Calc Chance, pg. 156-157) gives that:

$$S_N = 1 - \frac{f_N}{N^2} \quad \text{where} \quad f_N = \sum_{\substack{d>1 \\ d \mid \prod_{p\le N} p}} [\tfrac{N}{d}]^2$$

Further, (Calc Chance, pg. 155):

$$\frac{f_N}{N^2} = 1 - \prod_{p \le N} \left(1 - \frac{1}{p^2}\right) + g(N),$$

where g(x) is some function with $\lim_{N\to\infty} g(N) = 0$

In conclusion, one sees that:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} (1 - \frac{f_N}{N^2})$$

$$= \lim_{N\to\infty} (1 - \left(1 - \prod_{p \le N} \left(1 - \frac{1}{p^2}\right) + g(N)\right)) = \prod_p (1 - \frac{1}{p^2})$$

A different question: what is the probability that a∈[1,N] is square-free as $N \to \infty$, i.e.

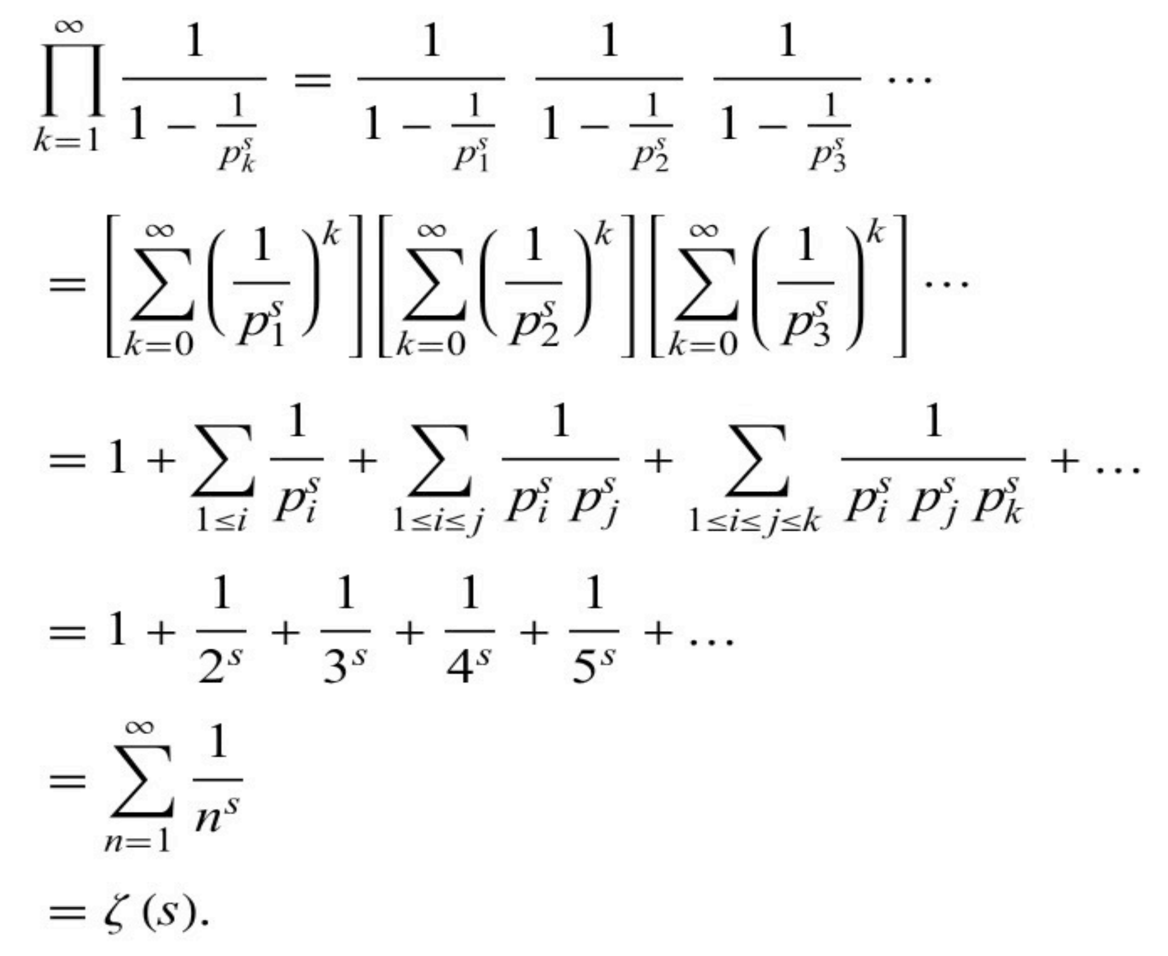
$$\lim_{N\to\infty} \frac{\#\{1 \leq a \leq N : p^2 \nmid a \text{ for all primes } p\}}{N}$$

Using the same ideas, one can show this limit is

$$\prod_p (1 - \frac{1}{p^2})$$

One estimates L:= $\prod_p \left(1-\frac{1}{p^2}\right) \approx .608$.

Is there a closed expression?



$$
\begin{aligned}
\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k^s}} &= \frac{1}{1-\frac{1}{p_1^s}}\ \frac{1}{1-\frac{1}{p_2^s}}\ \frac{1}{1-\frac{1}{p_3^s}} \cdots \\
&= \left[\sum_{k=0}^{\infty}\left(\frac{1}{p_1^s}\right)^k\right]\left[\sum_{k=0}^{\infty}\left(\frac{1}{p_2^s}\right)^k\right]\left[\sum_{k=0}^{\infty}\left(\frac{1}{p_3^s}\right)^k\right]\cdots \\
&= 1 + \sum_{1\le i} \frac{1}{p_i^s} + \sum_{1\le i\le j} \frac{1}{p_i^s\ p_j^s} + \sum_{1\le i\le j\le k} \frac{1}{p_i^s\ p_j^s\ p_k^s} + \ldots \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \zeta(s).
\end{aligned}
$$

Courtesy of
Wolfram Math World

This is the Basel Sum, which was originally proposed in 1650, and solved in closed form in 1734 by Euler.

There are many ways to find its closed form, and we will showcase two of them using complex analysis.

Proof #1:
Weierstrass Factorization Theorem (Stein-Shakarchi, 4.1):

> If $\{a_n\}$ is a sequence of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, then there is an entire function f vanishing *exactly* at $\{a_n\}$, and any other such function is of form $f(z)\, e^{g(z)}$, where g is also entire.

In particular, this gives that
$$\frac{sin(\pi z)}{\pi} = e^{g(z)} \cdot z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

for some entire function
$$g(z) = \sum_{n=1}^{\infty} g_n\, z^n$$

Next, we analyze the "order of growth" of g(z), which will give that g(z)=0.

The order of growth of a function f is

$$\rho(f) := \inf\{r : \exists A, B > 0 \; with \, |f| \leq A e^{B|z|^{r}}, \forall z \in \mathbb{C}\}$$

and one can show (Stein-Shakarchi 2.1) that if $\{z_i \neq 0\}$ are the zeroes of f(z), an entire function, and $\sum |z_i|^{-s} = \infty$, then $s \leq \rho(f)$.

Since $\sum_{n \neq 0} n^{-1} = \infty$, we see that $1 \leq \rho(\sin(\pi z))$.

But also, $|\sin(\pi z)| \leq e^{\pi|z|}$, so that $1 = \rho(\sin(\pi z))$.

If $g(z) = \sum_{n=1}^{\infty} g_n z^n$ with $g_n \neq 0$ for some $n > 1$, then $\rho(\sin(\pi z)) > 1$.

Therefore, g(z) is linear, and trying values forces g(z)=0.

Now, consider that

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = z - \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) z^3 + \cdots$$

but also,

$$\frac{\sin(\pi z)}{\pi} = \pi^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\pi z)^{2n+1}}{(2n+1)!} = z - \frac{\pi^2}{6} z^3 + \cdots$$

Proof #2:
We consider the function

$$B(z) := \frac{2\pi i}{z^2(e^{2\pi i z} - 1)}$$

noting it is analytic on $\mathbb{C}\backslash\mathbb{Z}$. Notice that $z^2B(z) = (z-n)^2B(z-n)$, so for each $n \in \mathbb{Z}$, we have a pole (of some order).

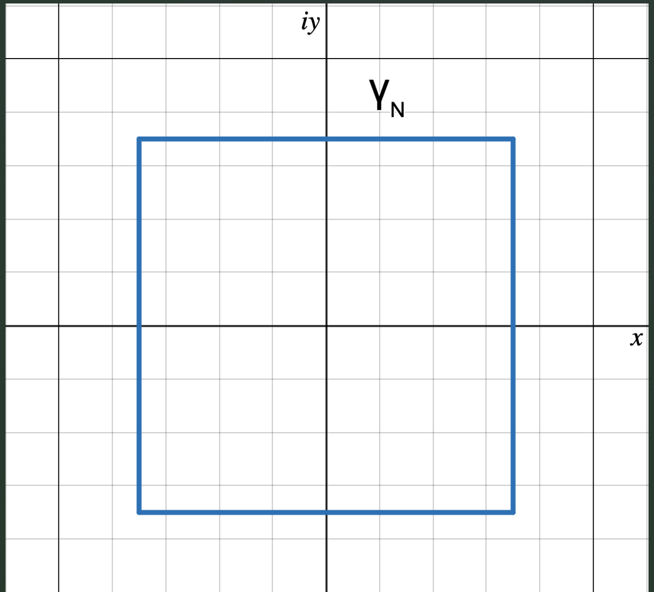
We will use the residue theorem on $B(z)$, which we will explore on the next slide.

The Residue Theorem (Stein-Shakarchi, 2.3):

If f is a holomorphic function on an open set with boundary given by a curve γ, except at poles $z_1,\ldots,z_n$ inside the open set, then

$$\oint_{\gamma} f(z)dz = 2\pi i \sum_{i=1}^{n} Res(f, z_i)$$

We will use this with the curve— a square of side length 2N+1— and then let N→∞.



Courtesy of Desmos

One can calculate $Res(B, 0)$ by noticing that if

$$B(z) = \sum_i b_i z^i ,$$

then since $2\pi i = z^2(e^{2\pi i z} - 1) \cdot \sum_i b_i z^i$,
also $Res(B, 0) = b_{-1} = -\frac{\pi^2}{3}$.

For $n \neq 0$, using $B(z) = \sum_i b_i (z-n)^i$, one gets

$$Res(B, n) = n^{-2}$$

Therefore, $2\sum_1^N \frac{1}{n^2} = \frac{\pi^2}{3} + \frac{1}{2\pi i}\oint_{\gamma_N} B(z)dz$.

By the Maximum Modulus Principle, B attains its maximum value on the boundary of γ_N. Therefore,

$$\int_{\gamma_N} B(z)dz \le \left(\max_{z\in\gamma_N} B(z)\right) * length(\gamma_N)$$

$$= \frac{8\pi\left(N+\frac{1}{2}\right)}{N^2(1-e^{-2\pi\left(N+\frac{1}{2}\right)})} \to 0, \quad \text{as } N \to \infty$$

And since $\frac{1}{2\pi i}\int_{\gamma_N} B(z)dz = -\frac{\pi^2}{3} + 2\sum_{n=1}^{N}\frac{1}{n^2}$,

we get

$$\sum_{i=1}^{\infty}\frac{1}{n^2} = \lim_{N\to\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

∎

One can check with that with the same analysis on the function,

$$B_k(z) := \frac{2\pi i}{z^{2k}(e^{2\pi i z} - 1)},$$

that

$$\zeta(2k) = \sum_n \frac{1}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!},$$

where B_j is the *j*-th Bernoulli number, defined by

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j ,$$

Oddly, B_j seems to pop up everywhere, and they are counting a natural combinatorial object (alternating permutations, i.e. “zigzag” numbers)

References

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