

Sum of Squares via the Theta Function

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Jacobi Theta Function:

Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$

The Theta function is defined for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ by:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

Two important special cases:

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \tau \in \mathbb{H}$$

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \quad t > 0$$

The nome:

$$q = e^{\pi i \tau}$$

is often used to make notation more compact, as

$$\theta(\tau) = \sum_n q^{n^2}$$

Also, for $\tau \in \mathbb{H}$, $|q(\tau)| < 1$, so convergence is no issue

Note: We will be using Complex Analysis by Stein/Shakarchi throughout

Properties of the Theta Function

The Theta function has several important properties that we will need to know:

1. Θ is entire in $z \in \mathbb{C}$ and holomorphic in $\tau \in \mathbb{H}$
2. $\Theta(z + 1|\tau) = \Theta(z|\tau)$
3. $\Theta(z + \tau|\tau) = \Theta(z|\tau)e^{-\pi i\tau} e^{-2\pi iz}$
4. $\Theta(z|\tau) = 0$ whenever $z = \frac{1}{2} + \frac{\tau}{2} + n + m\tau$ and $n, m \in \mathbb{Z}$

The periodicity shown in Property 2 means that this function has a Fourier Expansion

Introduction to Fourier Series/Transform

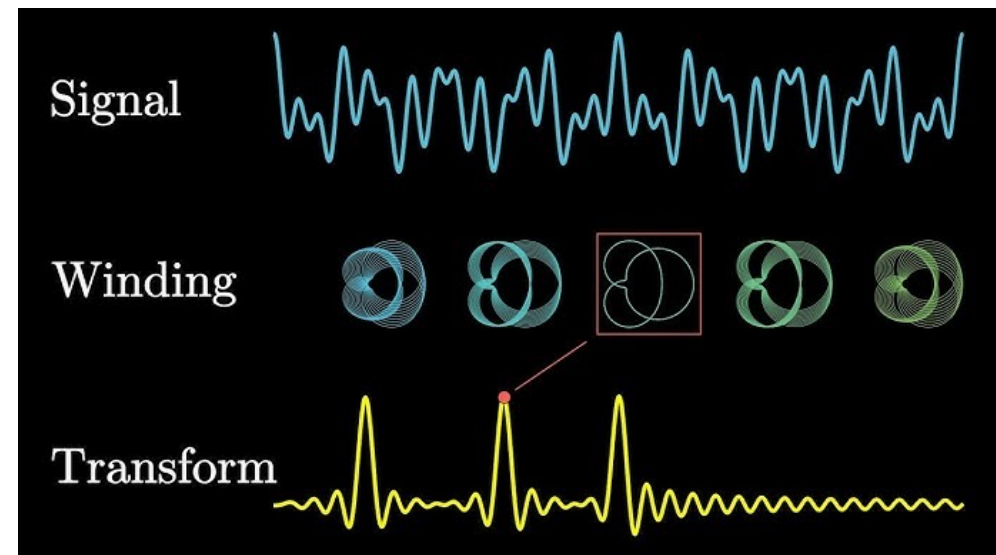
- Fourier transform: for function f

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{\pi i y x} dx$$

- Fourier series: if $f(x) = f(x+1)$ for x , then

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) q^n$$

- Fourier series are used to decompose complex signals into their much simpler parts
- They are often used in physics, engineering, and data compression



Properties of Fourier Series/Transform

- Gaussian is its own Fourier Transform:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$$

- If one applies Poisson Summation Formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

to the Gaussian, you get θ back, plus a further transformation property:

$$\text{If } \text{Im}(\tau) > 0, \text{ then } \theta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \theta(\tau)$$

Jacobi triple-product formula

- Defined in $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$:

$$\Pi(z|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2\pi iz})(1 + q^{2n-1}e^{-2\pi iz})$$

- We will then use Liouville's Theorem: "if f is entire and bounded, then f is constant,"
- Set $f := \theta/\Pi$, and consider its zeroes and poles to conclude f is constant. We find that constant to be 1, so $\theta = \Pi$
- This same approach (considering zeroes and poles) will be used later to prove our main theorems

Problem: Sums of 2 or 4 squares

- What numbers can be written as the sum of 2 or 4 squares of integers?

Ex) $5=1^2+2^2$, but 6 can't be written as a sum of 2 squares

But: $6=1^2+1^2+2^2+0^2$ (sum of 4 squares)

- Can we make this problem quantitative?
 - Let $r_2(n) := \#\{(a, b) \in \mathbb{Z}^2: a^2 + b^2 = n\}$ be the number of ways n can be written as the sum of 2 squares
 - $r_4(n)$ is similar for 4 squares
- We will use Fourier series, a better version of Liouville's THM, and generating functions to answer these questions.
- We *could* use the Triple Product formula and differentiation, but it is messy and unclear why it works

Generating Function

- For a function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ will define a generating function as:

$$G(f)(\tau) := \sum_{n \geq 0} f(n)q^n$$

- 2-squares theorem is equivalent to analyzing θ^2
- This is because:

$$\theta(\tau)^2 = \left(\sum_{n_1=-\infty}^{\infty} q^{n_1^2} \right) \left(\sum_{n_2=-\infty}^{\infty} q^{n_2^2} \right) = \sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} q^{n_1^2 + n_2^2} = \sum_{n=0}^{\infty} r_2(n)q^n$$

- Similarly, the 4-squares theorem is equivalent to analyzing θ^4

Proofs

- First, we will prove that:

$$r_2(n) = 4(d_1(n) - d_3(n))$$

where $d_1(n)$ denotes the number of divisors of n of the form $4k+1$, and $d_3(n)$ the number of divisors of n of the form $4k+3$

- Ex) $r_2(5) = 4(d_1(5) - d_3(5))$, $d_1(5) = 2$, $d_3(5) = 0$
- And $r_2(5) = 4(2 - 0) = 8$ since: $(1,2), (2,1), (-1,2), (2,-1), (1,-2), (-2,1), (-1,-2), (-2,-1) = 8$ possible ways
- The generating function identity $r_2(n)$ we want is thence:

$$G(r_2(n))(\tau) = \theta^2 = G(4(d_1(n) - d_3(n)))(\tau)$$

Proofs

- We now will find the generating function for $d_1(n)$:

$$\sum_{n=1}^{\infty} \frac{q^n}{1 - q^{4n}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q^{n(4m+1)} = \sum_{k=1}^{\infty} d_1(k)q^k,$$

because $d_1(k)$ counts the number of divisors of k that are of the form $4m + 1$. Observe that the series $\sum d_1(k)q^k$ converges since $d_1(k) \leq k$.

- $d_3(n)$ is found similarly:

$$\sum_{n=1}^{\infty} \frac{q^{3n}}{1 - q^{4n}} = \sum_{k=1}^{\infty} d_3(k)q^k,$$

- We can combine these to find:

$$G(4(d_1 - d_3)) = \sum_n \frac{1}{\cos(n\pi\tau)}$$

Proofs

- We will define $f = \frac{\theta^2}{G(4(d_1 - d_3))}$

Using a “fancy” version of Liouville’s THM, there are three properties needed to show f is constant:

1. $f(\tau + 2) = f(\tau)$
2. $f\left(-\frac{1}{\tau}\right) = f(\tau)$
3. $f(\tau)$ is bounded

- These properties are easy to check by looking at the zeros and poles of θ and of

$$G(4(d_1 - d_3)) = \sum_n \frac{1}{\cos(n\pi\tau)}$$

- So: f is constant, and we check that the constant is 1 by plugging in some value
- Therefore, $G(4(d_1 - d_3)) = \theta^2$, as desired.

Proofs

- The proof for the four squares theorem uses the same principles, but some new identities
- If $\sigma_1(r)$ denotes the sum of the positive divisors of r
- We can write its generating function as:

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{r=1}^{\infty} \sigma_1(r)q^r$$

- Similarly, if σ_1^* is defined as the sum of those divisors of n that are not divisible by 4
- We can write its generating function as:

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} - 4 \sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} = \sum_{r=1}^{\infty} \sigma_1^*(r)q^r.$$

- By using the previous variant of Liouville's THM, we can then find the generating function of r_4 to be:

$$G(r_4) = \theta^4 = G(4\sigma_1^*)$$

Sources

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